APG
Bresenham's Line Algorithm

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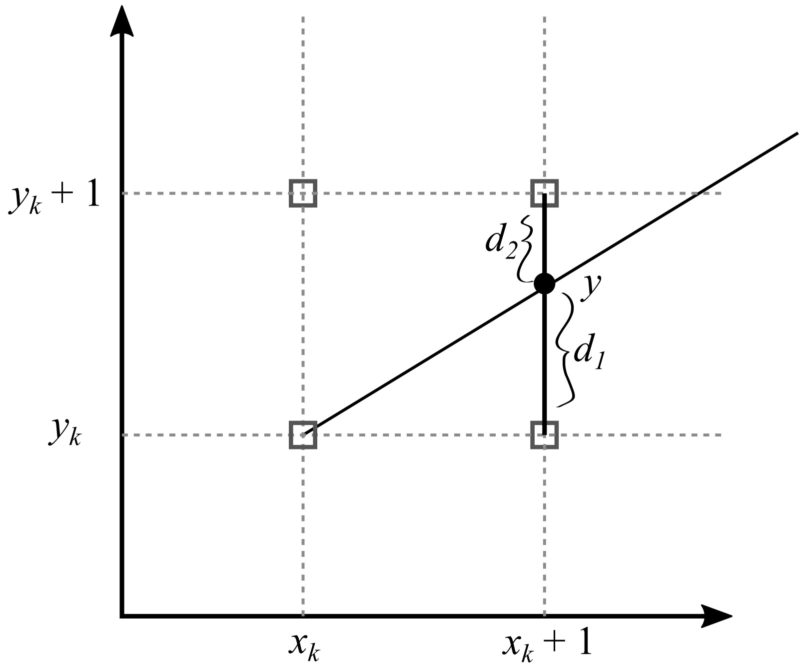

Figure 1: Bresenham line parameters when x is the driving axis.

Let us have a slope-intercept form of a line as

$$y = mx + b \tag{1}$$

The slope m can be obtained as

$$m = \frac{y_{k+1} - y_k}{x_{k+1} - x_k} = \frac{\Delta y}{\Delta x} \tag{2}$$

Beware, that y_{k+1} and $y_k + 1$ are a different thing!

Let the *driving axis* be x when $|m| \leq 1$ ($|\Delta x| \geq |\Delta y|$) or y when $|m| > 1$ ($|\Delta y| > |\Delta x|$). When the driving axis is x, it holds that $x_{k+1} = x_k + 1$. Similarly, when the driving axis is y, it holds that $y_{k+1} = y_k + 1$. Let us look at the case when the driving axis is x as shown in Figure 1. It is obvious that if $d_1 \leq d_2$, the next y value will be y_k, otherwise, if $d_1 > d_2$, next y will be $y_k + 1$. Let

$$\Delta d = d_1 - d_2 \tag{3}$$

From this, we can say that if $\Delta d \leq 0$, then $y = y_k$, while if $\Delta d > 0$, then $y = y_k + 1$. Therefore, we only need to know the sign of Δd to determine where the next pixel will be. Using the parameters as in Figure 1, the distances d_1 and d_2 can be expressed as

$$d_1 = y - y_k \tag{4}$$
$$d_2 = y_k + 1 - y \tag{5}$$

Since y lies on the line which is described with the equation $y = mx + b$ where x is currently $x_k + 1$, we can substitute in Equation 5 for y as $y = m(x_k + 1) + b$. This gives us

$$d_1 = m(x_k + 1) + b - y_k \tag{6}$$

$$d_2 = y_k + 1 - \big(m(x_k + 1) + b\big) = y_k + 1 - m(x_k + 1) - b \tag{7}$$

$$\tag{8}$$

Therefore

$$\begin{aligned} \Delta d = d_1 - d_2 &= \big(m(x_k + 1) + b - y_k\big) - \big(y_k + 1 - m(x_k + 1) - b\big) \\ &= m(x_k + 1) + b - y_k - y_k - 1 + m(x_k + 1) + b \\ &= 2m(x_k + 1) - 2y_k + 2b - 1 \end{aligned} \tag{9}$$

Now, since Bresenham's algorithm's main strength is the fact that it only uses basic arithmetic operations and comparisons and **integers**, we want to get rid of all floating point numbers in Δd. For this to be possible, it must hold that the algorithm gets four values as arguments, namely two points as follows: (x_1, y_1) and (x_2, y_2) where all four values are **rounded** and the two points are sorted from left to right, i.e. $x_1 < x_2$. Furthermore, as will be shown, b is not explicitely given which means that it is obtained from the four rounded points, and therefore it is also not problematic (it is not a floating point number). Additionally, b will be removed from the equations later on. The only problem is the value of $m = \Delta y / \Delta x$ which can be a floating point value in many cases even though Δy and Δx are integers. However, using a very simple and clever trick, we can obtain an equation with only integer values. We multiply both sides of Equation 9 with Δx. We obtain

$$\begin{aligned} \Delta x \Delta d &= \Delta x \Big(2y_k - 2m(x_k + 1) - 2b + 1\Big) \\ &= \Delta x \Big(2\frac{\Delta y}{\Delta x}(x_k + 1) - 2y_k + 2b - 1\Big) \\ &= 2\Delta y(x_k + 1) - 2\Delta x y_k + 2\Delta x b - \Delta x \\ &= 2\Delta y x_k - 2\Delta x y_k + \underbrace{2\Delta y + 2\Delta x b - \Delta x}_{\text{will be eliminated}} \end{aligned} \tag{10}$$

Let us say that p_k is a *prediction value* for step k:

$$p_k = \Delta x \Delta d \tag{11}$$

The magic of Bresenham's algorithm is that we can determine a prediction for next step p_{k+1} using previous prediction and simple arithmetic. Prediction for the next step is in general:

$$p_{k+1} = 2\Delta y x_{k+1} - 2\Delta x y_{k+1} + 2\Delta y + 2\Delta x b - \Delta x \tag{12}$$

As mentioned before, our goal is to compute p_{k+1} from previous step, i.e. p_k. Let us consider their difference:

$$\begin{aligned} p_{k+1} - p_k &= 2\Delta y x_{k+1} - 2\Delta x y_{k+1} + \underbrace{2\Delta y + 2\Delta x b - \Delta x}_{\text{const.}} - \big(2\Delta y x_k - 2\Delta x y_k + \underbrace{2\Delta y + 2\Delta x b - \Delta x}_{\text{const.}}\big) \\ &= 2\Delta y x_{k+1} - 2\Delta x y_{k+1} + \cancel{2\Delta y + 2\Delta x b - \Delta x} - \big(2\Delta y x_k - 2\Delta x y_k + \cancel{2\Delta y + 2\Delta x b - \Delta x}\big) \\ &= 2\Delta y x_{k+1} - 2\Delta x y_{k+1} - \big(2\Delta y x_k - 2\Delta x y_k\big) \\ &= 2\Delta y x_{k+1} - 2\Delta x y_{k+1} - 2\Delta y x_k + 2\Delta x y_k \end{aligned} \tag{13}$$

Remember, that because the driving axis is x, it holds that $x_{k+1} = x_k + 1$, therefore:

$$p_{k+1} - p_k = 2\Delta y(x_k + 1) - 2\Delta x y_{k+1} - 2\Delta y x_k + 2\Delta x y_k \tag{14}$$

Let us consider two cases: one where the current prediction $p_k \geq 0$ and second where $p_k < 0$.

Case 1: $p_k \geq 0$ When $p_k \geq 0$, it must hold that $\Delta d = d_1 - d_2 \geq 0$ since Δx is assumed to be positive (because the points are sorted from left to right). Therefore, it holds that $y_{k+1} = y_k$. When substituted into Equation 13 we obtain:

$$\begin{aligned} p_{k+1} - p_k &= 2\Delta y(x_k + 1) - 2\Delta x y_k - 2\Delta y x_k + 2\Delta x y_k \\ &= 2\Delta y x_k + 2\Delta y - 2\Delta x y_k - 2\Delta y x_k + 2\Delta x y_k \\ &= \cancel{2\Delta y x_k} + 2\Delta y - \cancel{2\Delta x y_k} - \cancel{2\Delta y x_k} + \cancel{2\Delta x y_k} \\ &= 2\Delta y \end{aligned} \tag{15}$$

Therefore, the next prediction can be defined as:

$$p_{k+1} = p_k + \underbrace{2\Delta y}_{\text{const.}} \tag{16}$$

Case 2: $p_k < 0$ When $p_k < 0$ it must hold that $\Delta d = d_1 - d_2 < 0$. Therefore, it holds that $y_{k+1} = y_k + 1$. Similarly to first case, we substitute y_{k+1} and obtain:

$$\begin{aligned} p_{k+1} - p_k &= 2\Delta y(x_k + 1) - 2\Delta x(y_k + 1) - 2\Delta y x_k + 2\Delta x y_k \\ &= 2\Delta y x_k + 2\Delta y - 2\Delta x y_k - 2\Delta x - 2\Delta y x_k + 2\Delta x y_k \\ &= \cancel{2\Delta y x_k} + 2\Delta y - \cancel{2\Delta x y_k} - 2\Delta x - \cancel{2\Delta y x_k} + \cancel{2\Delta x y_k} \\ &= 2\Delta y - 2\Delta x \end{aligned} \tag{17}$$

Therefore, the next prediction can be defined as:

$$p_{k+1} = p_k + \underbrace{2\Delta y - 2\Delta x}_{\text{const.}} \tag{18}$$

The two cases can be merged into a general equation as follows:

$$p_{k+1} = p_k + 2\Delta y - 2\Delta x(y_{k+1} - y_k) \tag{19}$$

where $y_{k+1} - y_k = 0$ for $p_k \geq 0$ and $y_{k+1} - y_k = 1$ for $p_k < 0$.